

Crystal Bases and Monomials for $U_q(G_2)$ -modules

Dong-Uy Shin *

School of Mathematics
Korea Institute for Advanced Study
Seoul 130-722, Korea
shindong@kias.re.kr

Abstract

In this paper, we give a new realization of crystal bases for irreducible highest weight modules over $U_q(G_2)$ in terms of monomials. We also discuss the natural connection between the monomial realization and tableau realization.

Introduction

In 1985, the *quantum groups* $U_q(\mathfrak{g})$, which may be thought of as q -deformations of the universal enveloping algebras $U(\mathfrak{g})$ of Kac-Moody algebras \mathfrak{g} , were introduced independently by Drinfel'd and Jimbo [1, 4]. The integrable highest weight representations over symmetrizable Kac-Moody algebras can be deformed consistently to the highest weight representations over the corresponding quantum groups for generic q [20]. From this point of view, the *crystal basis theory* for integrable modules over quantum groups was developed by Kashiwara [10, 11]. Crystal bases can be viewed as bases at $q = 0$ and they are given a structure of colored oriented graphs, called the *crystal graphs*. Crystal graphs have many nice combinatorial properties reflecting the internal structure of integrable modules. In [13], Kashiwara and Nakashima gave an explicit realization of crystal bases of finite dimensional irreducible modules over classical Lie algebras using semistandard tableaux with given shapes satisfying certain additional conditions. Motivated by their work, Kang and Misra discovered a tableau realization of crystal bases for finite dimensional irreducible modules over the exceptional Lie algebra G_2 [9]. In [17], Littelmann gave another description of crystal bases for finite dimensional simple Lie algebras using the Lakshmibai-Seshadri monomial theory. His approach was generalized to the *path model theory* for all symmetrizable Kac-Moody algebras [18, 19].

*This research was supported by KOSEF Grant # 98-0701-01-5-L

In [14], Kashiwara and Saito gave a geometric realization of the crystal graph $B(\infty)$ of $U_q^-(\mathfrak{g})$ as the set of irreducible components of a lagrangian subvariety \mathcal{L} of the quiver variety \mathfrak{M} and in [25], Saito extended their idea to the crystal base $B(\lambda)$ of irreducible highest weight modules of $U_q(\mathfrak{g})$. In [21], a crystal structure on the set of irreducible components of a lagrangian subvariety \mathfrak{Z} of the quiver variety \mathfrak{M} was given. In [22], Nakajima gave the set of $U_q(\mathbf{L}\mathfrak{g})$ -modules $M(P)$, called the *standard modules*, with a loop algebra $\mathbf{L}\mathfrak{g}$ of \mathfrak{g} and Drinfel'd polynomials P , and in [23] he introduced the t -analogs $\chi_{q,t}$ of q -character. While studying the t -analogs of q -character of standard modules M , he discovered that these irreducible components of a lagrangian subvariety \mathfrak{Z} are identified with certain monomials, and so the action of Kashiwara operators can be interpreted as multiplication by monomials. Moreover, in [12, 24], Kashiwara and Nakajima gave a crystal structure on the set \mathcal{M} of monomials and they showed that the connected component $\mathcal{M}(\lambda)$ of \mathcal{M} containing a highest weight vector M with a dominant integral weight λ is isomorphic to the irreducible highest weight crystal $B(\lambda)$. But, they did not give an explicit characterization of monomials in $\mathcal{M}(\lambda)$. For the special linear Lie algebra, in [8], Kang, Kim and the Author gave an explicit characterization and investigated the connection with the realization in terms of semistandard tableaux. Moreover, for the affine Lie algebra $A_n^{(1)}$, in [15], Kim gave an explicit description and she gave crystal isomorphisms from monomial realization to path realization and Young wall realization given in [3, 5, 6, 7].

In this paper, for any dominant integral weight λ , we give an explicit description of the crystal $\mathcal{M}(\lambda)$ for $U_q(G_2)$. In addition, we discuss the connection between the monomial realization and tableau realization of crystal bases given by Kang and Misra.

Acknowledgments. The author would like to express his sincere gratitude to Professor S.-J. Kang and Doctor J.-A. Kim for their interest in this work and many valuable discussions.

1 Crystal bases and Nakajima's monomial

1.1 Crystal bases for $U_q(G_2)$

Let $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ be a Cartan matrix of type G_2 . Then we have the finite dimensional simple Lie algebra G_2 and the *quantum group* $U_q(G_2)$ associated to A with a *Cartan subalgebra* \mathfrak{h} , the set of *simple roots* $\Pi = \{\alpha_1, \alpha_2\}$ and the set of *simple coroots* $\Pi^\vee = \{h_1, h_2\}$. Let $\Lambda_i \in \mathfrak{h}^*$ ($i = 1, 2$) be the *fundamental weight* and set $P = \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\Lambda_2$ and $P^+ = \mathbf{Z}_{\geq 0}\Lambda_1 \oplus \mathbf{Z}_{\geq 0}\Lambda_2$. Then every finite dimensional $U_q(G_2)$ -module M is a direct sum of irreducible modules $V(\lambda)$ with $\lambda \in P^+$.

Let M be a finite dimensional $U_q(\mathfrak{g})$ -module, then we have the *crystal basis* (L, B) [10, 11]. The set B gives a colored oriented graph structure with the arrow defined by

$$b \xrightarrow{i} b' \quad \text{if and only if} \quad \tilde{f}_i b = b'.$$

The graph B is called the *crystal graph* of M . Moreover, the crystal bases have a nice behavior with respect to the tensor product. See [2] for more details of quantum groups and crystal bases.

Now we give the description of the set of tableaux of type G_2 which is a realization of crystal basis [9]. Let λ be a dominant integral weight. Then the crystal graph $B(\lambda)$ of the irreducible highest weight module $V(\lambda)$ is realized as the set of tableaux on $I = \{1, 2, 3, 0, \bar{3}, \bar{2}, \bar{1}\}$ with the linear order $1 \prec 2 \prec 3 \prec 0 \prec \bar{3} \prec \bar{2} \prec \bar{1}$. To describe $B(\lambda)$, we need some definitions and conditions.

Definition 1.1. For $i, j = 0, 1, 2, 3$, we define

$$\begin{aligned} \text{dist}(i, j) &= \text{dist}(\bar{j}, \bar{i}) = j - i && \text{for } i < j, \ i, j = 1, 2, 3, \\ \text{dist}(i, 0) &= \text{dist}(0, \bar{i}) = 4 - i && \text{for } i = 1, 2, 3, \\ \text{dist}(i, \bar{j}) &= 8 - (i + j) && \text{for } i, j = 1, 2, 3. \end{aligned}$$

We also define $\text{dist}(a, b) = \text{dist}(b, a)$ and $\text{dist}(a, a) = 0$ for all $a, b \in I$.

Now, we are ready to give a characterization of tableaux for type $U_q(G_2)$.

Proposition 1.2. *For a dominant integral weight λ , the crystal graph $B(\lambda)$ is realized as the set of tableaux T of shape λ with entries on I such that*

- (i) *the entries of T weakly increase along the rows, but the element 0 cannot appear more than once,*
- (ii) *the entries of T strictly increase down the columns, but the element 0 can appear more than once,*
- (iii) *for each column C of length 2 with the entries a, b , $\text{dist}(a, b) \leq 2$ for $a = 1, 0$ and $\text{dist}(a, b) \leq 3$ otherwise,*
- (iv) *for each pair of adjacent columns C, C' of length 2 with the entries (from left to right and from top to bottom) a, b, c, d , $\text{dist}(a, d) \geq 3$ for $a = 2, 3, 0$ and $\text{dist}(a, d) \geq 2$ for $a = \bar{3}$.*

1.2 Nakajima's monomials

In this subsection, we recall the crystal structure on the set of monomials discovered by Nakajima [24]. Our presentation follows that of Kashiwara [12]. Moreover, we only treat the monomials for the type $U_q(G_2)$.

Let \mathcal{M} be the set of monomials in the variables $Y_i(n)$ for $i = 1, 2$ and $n \in \mathbf{Z}$. Here, a typical elements M of \mathcal{M} has the form $M = Y_{i_1}(n_1)^{a_1} \cdots Y_{i_r}(n_r)^{a_r}$, where $i_k = 1, 2$ and

$n_k, a_k \in \mathbf{Z}$ for $k = 1, \dots, r$. Since $Y_i(n)$'s are commuting variables, we may assume that $n_1 \leq n_2 \leq \dots \leq n_r$.

For a monomial $M = Y_{i_1}(n_1)^{a_1} \dots Y_{i_r}(n_r)^{a_r}$, we define

$$\begin{aligned} \text{wt}(M) &= \sum_{k=1}^r a_k \Lambda_{i_k} = a_1 \Lambda_{i_1} + \dots + a_r \Lambda_{i_r}, \\ \varphi_i(M) &= \max \left(\left\{ \sum_{\substack{k=1 \\ i_k=i}}^s a_k \mid 1 \leq s \leq r \right\} \cup \{0\} \right), \\ \varepsilon_i(M) &= \max \left(\left\{ - \sum_{\substack{k=s+1 \\ i_k=i}}^r a_k \mid 1 \leq s \leq r-1 \right\} \cup \{0\} \right). \end{aligned}$$

It is easy to verify that $\varphi_i(M) \geq 0, \varepsilon_i(M) \geq 0$, and $\langle h_i, \text{wt}(M) \rangle = \varphi_i(M) - \varepsilon_i(M)$.

First, we define

$$\begin{aligned} n_f &= \text{smallest } n_s \text{ such that } \varphi_i(M) = \sum_{\substack{k=1 \\ i_k=i}}^s a_k \\ n_e &= \text{largest } n_s \text{ such that } \varphi_i(M) = \sum_{\substack{k=1 \\ i_k=i}}^s a_k. \end{aligned}$$

In addition, choose integers c_{12} and c_{21} such that $c_{12} + c_{21} = 1$, and define

$$\begin{aligned} A_1(n) &= Y_1(n)Y_1(n+1)Y_2(n+c_{21})^{-1}, \\ A_2(n) &= Y_2(n)Y_2(n+1)Y_1(n+c_{12})^{-3}. \end{aligned}$$

Now, the *Kashiwara operators* \tilde{e}_i, \tilde{f}_i ($i = 1, 2$) on \mathcal{M} are defined as follows.

$$\begin{aligned} \tilde{f}_i(M) &= \begin{cases} 0 & \text{if } \varphi_i(M) = 0, \\ A_i(n_f)^{-1}M & \text{if } \varphi_i(M) > 0, \end{cases} \\ \tilde{e}_i(M) &= \begin{cases} 0 & \text{if } \varepsilon_i(M) = 0, \\ A_i(n_e)M & \text{if } \varepsilon_i(M) > 0. \end{cases} \end{aligned}$$

Then the maps $\text{wt} : \mathcal{M} \rightarrow P$, $\varphi_i, \varepsilon_i : \mathcal{M} \rightarrow \mathbf{Z} \cup \{-\infty\}$, $\tilde{e}_i, \tilde{f}_i : \mathcal{M} \rightarrow \mathcal{M} \cup \{0\}$ define a $U_q(\mathfrak{g})$ -crystal structure on \mathcal{M} [12].

Moreover, we have

Proposition 1.3. [12]

- (a) For each $i = 1, 2$, \mathcal{M} is isomorphic to a crystal graph of an integrable $U_{(i)}$ -module.
- (b) Let M be a monomial of weight λ such that $\tilde{e}_i M = 0$ for all $i = 1, 2$, and let $\mathcal{M}(\lambda)$ be the connected component of \mathcal{M} containing M . Then there exists a crystal isomorphism

$$\mathcal{M}(\lambda) \xrightarrow{\sim} B(\lambda) \text{ given by } M \mapsto v_\lambda.$$

Example 1.4. Let $c_{12} = 1$ and $c_{21} = 0$. Then the connected component containing $Y_1(0)$ which is isomorphic to the crystal graph $\mathcal{M}(\Lambda_1)$ is given as follows:

$$\begin{aligned} Y_1(0) &\xrightarrow{1} Y_1(1)^{-1}Y_2(0) \xrightarrow{2} Y_1(1)^2Y_2(1)^{-1} \xrightarrow{1} Y_1(1)Y_1(2)^{-1} \\ &\xrightarrow{1} Y_1(2)^{-2}Y_2(1) \xrightarrow{2} Y_1(2)Y_2(2)^{-1} \xrightarrow{1} Y_1(3)^{-1} \end{aligned}$$

2 Characterization of $\mathcal{M}(\lambda)$ and connection with tableau realization

2.1 Characterization of $\mathcal{M}(\lambda)$

In this subsection, we give an explicit characterization of the crystal $\mathcal{M}(\lambda)$ for $U_q(G_2)$. For simplicity, we take integers $c_{12} = 1$ and $c_{21} = 0$. Then we have

$$A_1(n) = Y_1(n)Y_1(n+1)Y_2(n)^{-1}, \quad A_2(n) = Y_2(n)Y_2(n+1)Y_1(n+1)^{-3}.$$

To characterize $\mathcal{M}(\lambda)$, we first focus on the case that $\lambda = \Lambda_k$ ($k = 1, 2$). Let $M_0 = Y_k(m)$ for $m \in \mathbf{Z}$, then we see that

$$\text{wt}(M_0) = \Lambda_k, \quad \varphi_i(M_0) = \delta_{ik} \text{ and } \varepsilon_i(M_0) = 0 \text{ for all } i = 1, 2.$$

Hence $\tilde{e}_i M_0 = 0$ for all $i = 1, 2$ and the connected component containing M_0 is isomorphic to $B(\Lambda_k)$ over $U_q(G_2)$. For simplicity, we will take $M_0 = Y_k(1)$, and that does not make much difference.

We set $Y_0(m)^{\pm 1} = 1$ for all $m \in \mathbf{Z}$. For $m \in \mathbf{Z}$, we introduce new variables

$$\begin{aligned} (2.1) \quad X_i(m) &= \begin{cases} Y_{i-1}(m+1)^{-1}Y_i(m) & \text{for } i = 1, 2, \\ Y_1(m+1)^2Y_2(m+1)^{-1} & \text{for } i = 3, \end{cases} \\ X_{\bar{i}}(m) &= \begin{cases} Y_{i-1}(m+(4-i))Y_i(m+(4-i))^{-1} & \text{for } i = 1, 2, \\ Y_1(m+2)^{-2}Y_2(m+1) & \text{for } i = 3, \end{cases} \\ X_0(m) &= Y_1(m+1)Y_1(m+2)^{-1}. \end{aligned}$$

Now, we give a lemma which plays a prominent role in characterizing the connected component containing a monomial M . By direct calculation, we have

Lemma 2.1. *For $m \in \mathbf{Z}$, we have*

$$\begin{aligned} (a) \quad X_1(m)X_0(m-1) &= X_2(m)X_3(m-1) & (b) \quad X_1(m)X_{\bar{3}}(m-1) &= X_2(m)X_0(m-1) \\ (c) \quad X_1(m)X_{\bar{2}}(m-1) &= X_3(m)X_0(m-1) & (d) \quad X_1(m)X_{\bar{1}}(m-1) &= X_0(m)X_0(m-1) \\ (e) \quad X_2(m)X_{\bar{2}}(m-1) &= X_3(m)X_{\bar{3}}(m-1) & (f) \quad X_2(m)X_{\bar{1}}(m-1) &= X_0(m)X_{\bar{3}}(m-1) \\ (g) \quad X_3(m)X_{\bar{1}}(m-1) &= X_0(m)X_{\bar{2}}(m-1) & (h) \quad X_0(m)X_{\bar{1}}(m-1) &= X_{\bar{3}}(m)X_{\bar{2}}(m-1) \end{aligned}$$

Remark 2.2. If we consider the monomials $X_\alpha(m)X_\beta(m-1)$ in the left hand side, we have

$$\begin{aligned} \text{dist}(\alpha, \beta) &\geq 3 \quad \text{if } \alpha = 2, 3 \text{ or } 0, \\ \text{dist}(\alpha, \beta) &\geq 2 \quad \text{if } \alpha = \bar{3}. \end{aligned}$$

Similarly, if we consider the monomials $X_\gamma(m)X_\delta(m-1)$ in the right hand side, we have

$$\begin{aligned} \text{dist}(\gamma, \delta) &\leq 2 \quad \text{if } \gamma = 1, 0, \\ \text{dist}(\gamma, \delta) &\leq 3 \quad \text{otherwise.} \end{aligned}$$

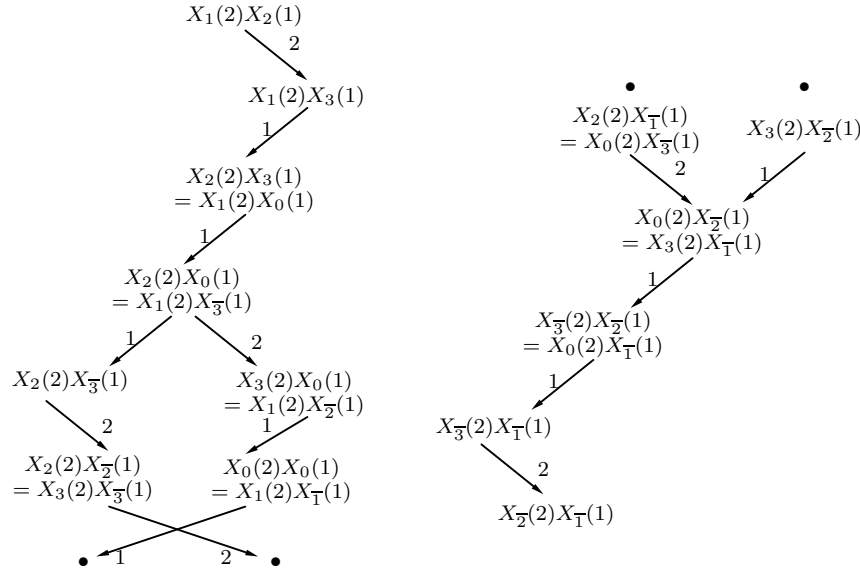
Proposition 2.3. (a) Let $M_0 = Y_1(1) = X_1(1)$ be a monomial of weight Λ_1 such that $\tilde{e}_i M_0 = 0$ for $i = 1, 2$, then the connected component $\mathcal{M}(\Lambda_1)$ of \mathcal{M} containing M_0 is

$$\{X_a(1) \mid a = 1, 2, 3, 0, \bar{3}, \bar{2}, \bar{1}\}.$$

(b) Let $M_0 = Y_2(1) = X_1(2)X_2(1)$ be a monomial of weight Λ_2 such that $\tilde{e}_i M_0 = 0$ for $i = 1, 2$. Then the connected component $\mathcal{M}(\Lambda_2)$ of \mathcal{M} containing M_0 is

$$\{X_a(2)X_b(1) \mid a \prec b, \text{ or } a = b = 0\}.$$

Proof. Since it is very easy to see (a), we only treat (b). By Proposition 1.3, it suffices to prove that $\mathcal{M}(\Lambda_2) \cup \{0\}$ is closed under the actions of \tilde{e}_i and \tilde{f}_i for $i = 1, 2$, and M_0 is the unique monomial of weight Λ_2 such that $\tilde{e}_i M_0 = 0$ for $i = 1, 2$. But, it is clear by the following connected component of $Y_2(1) = X_1(2)X_2(1)$.



□

Remark 2.4. If we take $M_0 = Y_k(N)$ ($k = 1, 2$), then we need to replace $X_i(m)$ by $X_i(m + N - 1)$.

Applying Lemma 2.1, i.e., replacing monomials in the left hand side with the monomials in the right hand side, we have

Theorem 2.5. Let $M_0 = Y_2(1) = X_1(2)X_2(1)$ be a monomial of weight Λ_2 such that $\tilde{e}_i M_0 = 0$ for $i = 1, 2$. Then the connected component $\mathcal{M}(\Lambda_2)$ of \mathcal{M} containing M_0 is characterized as

$$\mathcal{M}(\Lambda_2) = \left\{ X_a(2)X_b(1) \mid \begin{array}{l} \text{(i) } a \prec b, \text{ or } a = b = 0, \\ \text{(ii) } \text{dist}(a, b) \leq 2 \text{ for } a = 1, 0, \text{ dist}(a, b) \leq 3 \text{ otherwise.} \end{array} \right\}.$$

We now consider the general case. By direct calculation, we have

Lemma 2.6. For $m \in \mathbf{Z}$, we have

$$X_0(m)^2 = X_3(m)X_{\overline{3}}(m).$$

Proposition 2.7. Let $\lambda = m\Lambda_1 + n\Lambda_2$. Then the connected component $\mathcal{M}(\lambda)$ containing

$$M_0 = Y_1(1)^m Y_2(1)^n = X_1(1)^m (X_1(2)X_2(1))^n$$

is expressed as the set of monomials

$$M = X_{a_1}(2) \cdots X_{a_n}(2) X_{b_1}(1) \cdots X_{b_{m+n}}(1) \\ (a_1 \succeq \cdots \succeq a_n, b_1 \succeq \cdots \succeq b_{m+n})$$

satisfying following conditions:

$$(2.2) \quad a_j \prec b_j \quad \text{unless } a_j = b_j = 0 \text{ for } j = 1, \dots, n.$$

Proof. As in Proposition 2.3, it suffices to prove the following statements:

- (a) For all $i = 1, 2$, we have $\tilde{e}_i \mathcal{M}(\lambda) \subset \mathcal{M}(\lambda) \cup \{0\}$ and $\tilde{f}_i \mathcal{M}(\lambda) \subset \mathcal{M}(\lambda) \cup \{0\}$.
- (b) If $M \in \mathcal{M}(\lambda)$ and $\tilde{e}_i M = 0$ for all $i = 1, 2$, then $M = M_0$.

We first prove the statement (a). Let M be a monomial of $\mathcal{M}(\lambda)$. Assume that $\tilde{f}_i M \neq 0$ for $i = 1, 2$. Then $\tilde{f}_i M$ is obtained by multiplying $A_i(k)^{-1}$ from M for some k . Note that $A_1(k)^{-1}$ and $A_2(k)^{-1}$ are expressed as

$$(2.3) \quad \begin{aligned} A_1(k)^{-1} &= X_1(k)^{-1} X_2(k) = X_3(k-1)^{-1} X_0(k-1) \\ &= X_0(k-1)^{-1} X_{\overline{3}}(k-1) = X_{\overline{2}}(k-2)^{-1} X_{\overline{1}}(k-2) \end{aligned}$$

and

$$(2.4) \quad A_2(k)^{-1} = X_2(k)^{-1} X_3(k) = X_{\overline{3}}(k-1)^{-1} X_{\overline{2}}(k-1).$$

Suppose that $\tilde{f}_i M$ does not satisfy the condition (2.2). It means that

$$A_i(k)^{-1} = X_p(2)^{-1} X_{p+1}(2) \text{ for some } p \text{ determined by (2.3) and (2.4)}$$

and there is no element among b_1, \dots, b_n larger than $p+1$, where $p+1$ denote by the next element of p under the ordering in I . But, by the condition (2.2), there should be a $X_{p+1}(1)$ in M and so it is a contradiction.

Similarly, we can prove that $\tilde{e}_i \mathcal{M}(\lambda) \subset \mathcal{M}(\lambda) \cup \{0\}$.

To prove (b), suppose $M \in \mathcal{M}(\lambda)$ and $\tilde{e}_i M = 0$ for all $i = 1, 2$. Then we can easily check that $b_{n+1}, \dots, b_{m+n}, a_1, \dots, a_n$ should be 1. Moreover, b_1, \dots, b_n should be 2. Therefore, we have $M = X_1(1)^m (X_1(2)X_2(1))^n = Y_1(1)^m Y_2(1)^n$. \square

Remark 2.8. For dominant integral weights λ , μ , and τ such that $\lambda = \mu + \tau$, we have

$$\mathcal{M}(\lambda) = \{M_1 M_2 \mid M_1 \in \mathcal{M}(\mu), M_2 \in \mathcal{M}(\tau)\}.$$

Now, we consider a monomial M in $\mathcal{M}(\lambda)$ with a dominant integral weight $\lambda = m\Lambda_1 + n\Lambda_2$. At first, for any expression of

$$M = X_{a_1}(2) \cdots X_{a_n}(2) X_{b_1}(1) \cdots X_{b_{m+n}}(1),$$

we put $X_i(j)$ in the j -th row from bottom so that $X_b(j)$ exists to the right hand side of $X_a(j)$ for $a \preceq b$. That is,

$$\begin{array}{c} X_{a_n}(2) \cdots X_{a_1}(2) \\ X_{b_{n+m}}(1) \cdots X_{b_{n+1}}(1) X_{b_n}(1) \cdots X_{b_1}(1) \end{array}$$

Secondly, we apply the following rules:

- (al-1) If there is a pair $(X_\alpha(2), X_\beta(1))$ such that $X_\alpha(2)X_\beta(1)$ is one of monomials in the left hand side of Lemma 2.1, and $X_\alpha(2)$ and $X_\beta(1)$ lie in the same column or $X_\beta(1)$ lies in the left hand side of $X_\alpha(2)$, then we replace $(X_\alpha(2), X_\beta(1))$ with $(X_\gamma(2), X_\delta(1))$ which is the pair of monomials corresponding to $X_\alpha(2), X_\beta(1)$. Moreover, if there are several such pairs, then we apply above rule from the pair $(X_\alpha(2), X_\beta(1))$ with the largest distance between $X_\alpha(2)$ and $X_\beta(1)$ to the pair $(X_{\alpha'}(2), X_{\beta'}(1))$ with the smallest distance between $X_{\alpha'}(2)$ and $X_{\beta'}(1)$.
- (al-2) If there is a pair $(X_\gamma(2), X_\delta(1))$ such that $X_\gamma(2)X_\delta(1)$ is one of monomials in the right hand side of Lemma 2.1, and $X_\delta(1)$ lies in the right hand side of $X_\gamma(2)$, then we replace $(X_\gamma(2), X_\delta(1))$ with $(X_\alpha(2), X_\beta(1))$ which is the pair of monomials corresponding to $X_\gamma(2), X_\delta(1)$. Moreover, if there are several such pairs, then we apply above rule from the pair $(X_\gamma(2), X_\delta(1))$ with the largest distance between $X_\gamma(2)$ and $X_\delta(1)$ to the pair $(X_{\gamma'}(2), X_{\delta'}(1))$ with the smallest distance between $X_{\gamma'}(2)$ and $X_{\delta'}(1)$.
- (al-3) If there is a pair $(X_0(p), X_0(p))$ ($p = 1, 2$), then we replace $(X_0(p), X_0(p))$ by $(X_3(p), X_{\bar{3}}(p))$.

From now on, we denote by $[M]$ the expression of monomial M obtained by (al-1), (al-2) and (al-3).

Example 2.9. (a) Let λ be a dominant integral weight $\Lambda_1 + 3\Lambda_2$ and let M be a monomial $Y_1(2)^3 Y_1(3)^{-3} Y_1(4)^{-1} Y_2(2)^2 Y_2(3)^{-1}$, then it can be expressed as

$$M = X_2(2) X_1(2)^2 X_{\bar{1}}(1) X_{\bar{2}}(1) X_{\bar{3}}(1) X_0(1)$$

and so it is a monomial of $\mathcal{M}(\lambda)$. Moreover, we have

$$\begin{aligned}
X_1(2)X_1(2)X_2(2) &= X_1(2)X_2(2)X_2(2) \\
X_0(1)X_{\overline{3}}(1)X_{\overline{2}}(1)X_{\overline{1}}(1) &\stackrel{(\text{al-1})}{=} X_3(1)X_{\overline{3}}(1)X_{\overline{2}}(1)X_{\overline{1}}(1) \\
&= X_1(2)X_2(2)X_3(2) \\
&\stackrel{(\text{al-1})}{=} X_3(1)X_{\overline{3}}(1)X_{\overline{3}}(1)X_{\overline{1}}(1) \\
&= X_2(2)X_2(2)X_0(2) \\
&\stackrel{(\text{al-1})}{=} X_3(1)X_0(1)X_{\overline{3}}(1)X_{\overline{2}}(1) .
\end{aligned}$$

Therefore,

$$[M] = X_0(2)X_2(2)^2X_{\overline{2}}(1)X_{\overline{3}}(1)X_0(1)X_3(1).$$

(b) Let λ be a dominant integral weight $2\Lambda_2$ and let M be a monomial $Y_1(2)^2Y_1(4)^{-2}$, then it can be expressed as

$$M = X_0(2)^2X_0(1)^2$$

and so it is a monomial of $\mathcal{M}(\lambda)$. Moreover, we have

$$\begin{aligned}
X_0(2)X_0(2) &= X_1(2)X_0(2) = X_2(2)X_{\overline{3}}(2) \\
X_0(1)X_0(1) &\stackrel{(\text{al-2})}{=} X_0(1)X_{\overline{1}}(1) \stackrel{(\text{al-1})}{=} X_3(1)X_{\overline{2}}(1) .
\end{aligned}$$

Therefore,

$$[M] = X_{\overline{3}}(2)X_2(2)X_{\overline{2}}(1)X_3(1).$$

By the algorithm (al-1),(al-2) and (al-3), we have

Theorem 2.10. *Let $\lambda = m\Lambda_1 + n\Lambda_2$. Then the connected component $\mathcal{M}(\lambda)$ containing*

$$M_0 = Y_1(1)^mY_2(1)^n = X_1(1)^m(X_1(2)X_2(1))^n$$

is expressed as the set of monomials

$$\begin{aligned}
M &= X_{a_1}(2) \cdots X_{a_n}(2)X_{b_1}(1) \cdots X_{b_{m+n}}(1) \\
&\quad (a_1 \succeq \cdots \succeq a_n, \quad b_1 \succeq \cdots \succeq b_{m+n})
\end{aligned}$$

satisfying following conditions:

- (i) 0's can not be repeated in a_1, \dots, a_n and b_1, \dots, b_{m+n} , respectively,
- (ii) $a_j \prec b_j$ unless $a_j = b_j = 0$ for $j = 1, \dots, n$,
- (iii) for $j = 1, \dots, n$,

$$\begin{aligned}
\text{dist}(a_j, b_j) &\leq 2 \quad \text{if } a_j = 1 \text{ or } 0, \\
\text{dist}(a_j, b_j) &\leq 3 \quad \text{otherwise,}
\end{aligned}$$

- (iv) for $j = 2, \dots, n$,

$$\begin{aligned}
\text{dist}(a_j, b_{j-1}) &\geq 3 \quad \text{if } a_j = 2, 3 \text{ or } 0, \\
\text{dist}(a_j, b_{j-1}) &\geq 2 \quad \text{if } a_j = \overline{3}.
\end{aligned}$$

2.2 Connection with other realizations of crystal graphs

In this subsection, we give a natural 1-1 correspondence between monomial realization and tableau realization of crystal bases of type $U_q(G_2)$.

Let $\lambda = m\Lambda_1 + n\Lambda_2$ be a dominant integral weight. The tableau realization $T(\lambda)$ of $B(\lambda)$ given by Kang and Misra was obtained by imbedding $B(\lambda)$ to $B(\Lambda_1)^{\otimes m} \otimes B(\Lambda_2)^{\otimes n}$. Similarly, if we imbed $B(\lambda)$ to $B(\Lambda_2)^{\otimes n} \otimes B(\Lambda_1)^{\otimes m}$, we can have another tableau realization $S(\lambda)$. That is, for a dominant integral weight $\lambda = m\Lambda_1 + n\Lambda_2$, we can associate a diagram which is a collection of n boxes in top row and $m+n$ boxes in bottom row in right justified rows and $B(\lambda)$ is realized as the set $S(\lambda)$ of tableaux satisfying the same condition as $T(\lambda)$ on this diagram.

Theorem 2.11. *Let $\lambda = m\Lambda_1 + n\Lambda_2$ be a dominant integral weight. Then there is a crystal isomorphism $\psi : \mathcal{M}(\lambda) \rightarrow S(\lambda)$.*

Proof. Let M be a monomial in $\mathcal{M}(\lambda)$. Then M is expressed as

$$M = X_{a_1}(2) \cdots X_{a_n}(2) X_{b_1}(1) \cdots X_{b_{m+n}}(1)$$

in Theorem 2.10. We define $\psi(M)$ by the tableau

$$\begin{array}{|c|c|c|c|c|} \hline & & & a_n & \cdots & a_1 \\ \hline b_{m+n} & \cdots & b_{n+1} & b_n & \cdots & b_1 \\ \hline \end{array}.$$

Then it is clear that $\psi(M)$ belongs to $S(\lambda)$ by the condition (i)-(iv) in Theorem 2.10.

Conversely, for a tableau

$$T = \begin{array}{|c|c|c|c|c|c|} \hline & & & a_n & \cdots & a_1 \\ \hline b_{m+n} & \cdots & b_{n+1} & b_n & \cdots & b_1 \\ \hline \end{array} \in S(\lambda),$$

we also define $\psi^{-1}(T)$ by the monomial

$$X_{a_1}(2) \cdots X_{a_n}(2) X_{b_1}(1) \cdots X_{b_{m+n}}(1).$$

Then by definition it is clear that ψ and ψ^{-1} are inverses to each other.

Now, it remains to show that ψ is a crystal morphism. But, because of (2.3) and (2.4), it is easy to see that ψ is a crystal morphism of $U_q(G_2)$ -modules from the definition of Kashiwara operators on the set \mathcal{M} of monomials and the tensor product rule of Kashiwara operators which is applied to the set $S(\lambda)$. \square

Example 2.12. Let λ be a dominant integral weight $\Lambda_1 + 3\Lambda_2$ and let M be a monomial $Y_1(2)^3 Y_1(3)^{-3} Y_1(4)^{-1} Y_2(2)^2 Y_2(3)^{-1} \in \mathcal{M}(\lambda)$ given in Example 2.9 (a), then it can be expressed as

$$M = X_0(2) X_2(2)^2 X_{\bar{2}}(1) X_{\bar{3}}(1) X_0(1) X_3(1).$$

Then we have

$$\psi(M) = \begin{array}{|c|c|c|c|} \hline & 2 & 2 & 0 \\ \hline 3 & 0 & \bar{3} & \bar{2} \\ \hline \end{array}.$$

We have the following proposition between $S(\lambda)$ and $T(\lambda)$.

Proposition 2.13. [16] *For a dominant integral weight $\lambda = m\Lambda_1 + n\Lambda_2$, there is a crystal isomorphism $\varphi : S(\lambda) \rightarrow T(\lambda)$ for $U_q(G_2)$ -module given by*

$$\varphi(S) = S_{2,1} \leftarrow S_{2,2} \leftarrow \cdots \leftarrow S_{2,n} \leftarrow S_{1,1} \leftarrow \cdots \leftarrow S_{1,m},$$

where $S_{i,j} \in S(\Lambda_i)$ ($i = 1, 2$) is the column of S of length i from right to left.

Corollary 2.14. *Let $\lambda = m\Lambda_1 + n\Lambda_2$ be a dominant integral weight. There is a crystal isomorphism $\phi : \mathcal{M}(\lambda) \rightarrow T(\lambda)$.*

Proof. By Theorem 2.10 and Proposition 2.13, $\phi = \varphi \circ \psi$ is a crystal isomorphism. \square

Example 2.15. Let M be a monomial $Y_1(2)^3 Y_1(3)^{-3} Y_1(4)^{-1} Y_2(2)^2 Y_2(3)^{-1}$ given in Example 2.12. Then we have

$$\begin{aligned} \phi(M) &= S_{2,1} \leftarrow S_{2,2} \leftarrow S_{2,3} \leftarrow S_{1,1} \\ &= \begin{array}{|c|} \hline 0 \\ \hline \bar{2} \\ \hline \end{array} \leftarrow \begin{array}{|c|} \hline 2 \\ \hline \bar{3} \\ \hline \end{array} \leftarrow \begin{array}{|c|} \hline 2 \\ \hline 0 \\ \hline \end{array} \leftarrow \begin{array}{|c|} \hline 3 \\ \hline \end{array} \\ &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & \bar{1} \\ \hline 3 & \bar{3} & \bar{3} & \\ \hline \end{array}. \end{aligned}$$

Conversely, let T be a tableau of $T(\Lambda_1 + 3\Lambda_2)$

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & \bar{1} \\ \hline 3 & \bar{3} & \bar{3} & \\ \hline \end{array}.$$

By applying the reverse bumping rule to the entries from bottom to top and from right to left, i.e., from the entry $\bar{1}$ of the rightmost column to the entry 1 on top of the leftmost column, we have the following sequence

$$(0, \bar{2}, 2, \bar{3}, 2, 0, 3).$$

Therefore, we have

$$S_{2,1} = \begin{array}{|c|} \hline 0 \\ \hline \bar{2} \\ \hline \end{array}, \quad S_{2,2} = \begin{array}{|c|} \hline 2 \\ \hline \bar{3} \\ \hline \end{array}, \quad S_{2,3} = \begin{array}{|c|} \hline 2 \\ \hline 0 \\ \hline \end{array} \text{ and } S_{1,1} = \begin{array}{|c|} \hline 3 \\ \hline \end{array},$$

and since

$$\begin{aligned} \psi^{-1}(S_{2,1}) &= X_0(2)X_{\bar{2}}(1), \quad \psi^{-1}(S_{2,2}) = X_2(2)X_{\bar{3}}(1), \\ \psi^{-1}(S_{2,3}) &= X_2(2)X_0(1), \quad \psi^{-1}(S_{1,1}) = X_3(1), \end{aligned}$$

we have

$$\begin{aligned} \varphi^{-1}(T) &= \psi^{-1}(S_{2,1})\psi^{-1}(S_{2,2})\psi^{-1}(S_{2,3})\psi^{-1}(S_{1,1}) \\ &= Y_1(2)^3 Y_1(3)^{-3} Y_1(4)^{-1} Y_2(2)^2 Y_2(3)^{-1}. \end{aligned}$$

2.3 Another realization of crystal graphs

In this subsection, we discuss about another connected component which is isomorphic to the crystal graph $B(\lambda)$ with a dominant integral weight λ . More precisely, let M_0 be a monomial $Y_1(-1)^m Y_2(-2)^n = X_1(-1)^m (X_1(-1)X_2(-2))^n$. Then M_0 is a monomial of weight $m\Lambda_1 + n\Lambda_2$ such that $\tilde{e}_i M_0 = 0$ for all $i = 1, 2$. Moreover, we have

Theorem 2.16. *Let $\lambda = m\Lambda_1 + n\Lambda_2$. Then the connected component $\mathcal{M}'(\lambda)$ containing*

$$M_0 = Y_1(-1)^m Y_2(-2)^n = X_1(-1)^m (X_1(-1)X_2(-2))^n$$

is expressed as the set of monomials

$$M = X_{a_1}(-1) \cdots X_{a_{m+n}}(-1) X_{b_1}(-2) \cdots X_{b_n}(-2) \\ (a_1 \preceq \cdots \preceq a_{m+n}, b_1 \preceq \cdots \preceq b_n)$$

satisfying following conditions:

- (i) 0's can not be repeated in a_1, \dots, a_{m+n} and b_1, \dots, b_n , respectively,
- (ii) $a_j \prec b_j$ unless $a_j = b_j = 0$ for $j = 1, \dots, n$,
- (iii) for $j = 1, \dots, n$,

$$\begin{aligned} \text{dist}(a_j, b_j) &\leq 2 \quad \text{if } a_j = 1 \text{ or } 0, \\ \text{dist}(a_j, b_j) &\leq 3 \quad \text{otherwise,} \end{aligned}$$

- (iv) for $j = 1, \dots, n-1$,

$$\begin{aligned} \text{dist}(a_j, b_{j+1}) &\geq 3 \quad \text{if } a_j = 2, 3 \text{ or } 0, \\ \text{dist}(a_j, b_{j+1}) &\geq 2 \quad \text{if } a_j = \bar{3}. \end{aligned}$$

Proof. By the same argument in Proposition 2.7, it is proved. So we omit it. \square

Theorem 2.17. *Let $\lambda = m\Lambda_1 + n\Lambda_2$ be a dominant integral weight. Then there is a crystal isomorphism $\psi : \mathcal{M}'(\lambda) \rightarrow T(\lambda)$.*

Proof. Let M be a monomial in $\mathcal{M}'(\lambda)$. Then M is expressed as

$$M = X_{a_1}(-1) \cdots X_{a_{m+n}}(-1) X_{b_1}(-2) \cdots X_{b_n}(-2).$$

We define $\psi(M)$ by the tableau

$$\begin{array}{|c|c|c|c|c|c|} \hline a_1 & \cdots & a_n & a_{n+1} & \cdots & a_{m+n} \\ \hline b_1 & \cdots & b_n & & & \\ \hline \end{array}.$$

Then it is clear that $\psi(M)$ belongs to $T(\lambda)$ by the condition (i)-(iv) in Theorem 2.16.

Conversely, for a tableau

$$T = \begin{array}{|c|c|c|c|c|c|} \hline a_1 & \cdots & a_n & a_{n+1} & \cdots & a_{m+n} \\ \hline b_1 & \cdots & b_n & & & \\ \hline \end{array} \in T(\lambda),$$

we also define $\psi^{-1}(T)$ by the monomial

$$X_{a_1}(-1) \cdots X_{a_{m+n}}(-1) X_{b_1}(-2) \cdots X_{b_n}(-2).$$

Then by definition it is clear that ψ and ψ^{-1} are inverses to each other. Moreover, it is also easy to see that ψ is a crystal morphism of $U_q(G_2)$ -modules. \square

References

- [1] V. G. Drinfel'd, *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Dokl. **32** (1985), 254–258.
- [2] J. Hong, S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Graduate Studies in Mathematics **42**, Amer. Math. Soc., 2002.
- [3] J. Hong, S.-J. Kang, H. Lee, *Young wall realization of higher level crystal graphs for quantum affine algebras*, in preparation.
- [4] M. Jimbo, *A q -difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 63–69.
- [5] M. Jimbo, K. C. Misra, T. Miwa, and M. Okado, *Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}}(n))$ at $q = 0$* , Comm. Math. Phys. **136** (1991), 543–566.
- [6] S.-J. Kang, *Crystal bases for quantum affine algebras and combinatorics of Young walls*, Proc. London Math. Soc. **86** (2003), 29–69.
- [7] S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki, *Affine crystals and vertex models*, Int. J. Mod. Phys. A. **Suppl. 1A** (1992), 449–484.
- [8] S.-J. Kang, J.-A. Kim, D.-U. Shin, *Monomial realization of crystal bases for special linear Lie algebras*, preprint math.RT/0303232, to appear in J. Algebra.
- [9] S.-J. Kang, K. C. Misra, *Crystal bases and tensor product decompositions of $U_q(G_2)$ -modules*, J. Algebra **163** (1994), 675–691.
- [10] M. Kashiwara, *Crystalizing the q -analogue of universal enveloping algebras*, Comm. Math. Phys. **133** (1990), 249–260.
- [11] ———, *On crystal bases of the q -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.

- [12] ———, *Realizations of crystals*, Contemp. Math. **325** (2003), Amer. Math. Soc., 133–139.
- [13] M. Kashiwara, T. Nakashima, *Crystal graphs for representations of the q -analogue of classical Lie algebras*, J. Algebra **165** (1994), 295–345.
- [14] M. Kashiwara, Y. Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), 9–36.
- [15] J.-A. Kim, *Monomial realization of crystal graphs for $U_q(A_n^{(1)})$* , submitted.
- [16] C. Lecouvey, *Schensted type correspondence for type G_2 and computation of the canonical basis of a finite dimensional $U_q(G_2)$ -module*, preprint math.CO/0211443.
- [17] P. Littelmann, *Crystal graphs and Young tableaux*, J. Algebra **175** (1995), 65–87.
- [18] ———, *Paths and root operators in representation theory*, Ann. of Math. **142** (1995), 499–525.
- [19] ———, *A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras*, Invent. Math. **116** (1994), 329–346.
- [20] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.
- [21] H. Nakajima, *Quiver varieties and tensor products*, Invent. Math. **146** (2001), 399–449.
- [22] ———, *Quiver varieties and finite dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), 145–238.
- [23] ———, *Quiver varieties and t -analogs of q -characters of quantum affine algebras*, preprint math.QA/0105713.
- [24] ———, *t -analogs of q -characters of quantum affine algebras of type A_n , D_n* , Contemp. Math. **325** (2003), Amer. Math. Soc. 141–160.
- [25] Y. Saito, *Crystal bases and quiver varieties*, Math. Ann. **324** (2002), 675–688.